arXiv:hep-ph/0105264v1 25 May 2001

# Slope of the topological susceptibility at zero temperature and finite temperature in the Nambu–Jona-Lasinio model

K. Fukushima*, K. Ohnishi† and K. Ohta‡

*Institute of Physics, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8902, Japan*

We estimate the slope of the topological susceptibility in the three flavour Nambu–Jona-Lasinio model with the 't Hooft interaction. The results are consistent with the evaluation from the QCD sum rule in favour of the full topological susceptibility. We apply it to the Shore-Veneziano formula to find that it shows satisfactory agreement with the anomalous suppression of the flavour-singlet axial charge. The behaviour at finite temperature is also discussed.

PACS numbers: 11.30.Rd, 12.39.Fe

*1. Introduction.* It is known that the $\mathrm{U_A}(1)$ anomaly embodies one of the most essential implications in the non-perturbative QCD physics [1]. The mass of the $\eta'$ meson is significantly enhanced (called the "$\mathrm{U_A}(1)$ problem") and the flavour-singlet axial charge $a^0$ is highly suppressed (called the "spin problem") due to the anomalous breaking of the $\mathrm{U_A}(1)$ symmetry. Both phenomena are deeply connected with another quantity sensitive to the topological characteristics of QCD, that is, the topological susceptibility given by

$$\chi(k^2) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx} \langle 0|\mathrm{T}Q(x)Q(0)|0\rangle_{\mathrm{connected}}, \tag{1}$$

where $Q(x)$ is the topological charge density. The Witten-Veneziano mass formula [2] discloses the beautiful relation between the quantities concerning the $\mathrm{U_A}(1)$ sector, i.e.

$$\frac{2N_\mathrm{f}}{f_\pi^2}\chi_{\mathrm{pure}}(0) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2, \tag{2}$$

where $N_\mathrm{f} = 3$ is the number of the flavours and $f_\pi$ is the pion decay constant. This formula has been well established and actually confirmed by the lattice calculation, which gives $\chi_{\mathrm{pure}}(0) \sim (175\,\mathrm{MeV})^4$ [3] as compared with the phenomenological value $\sim (180\,\mathrm{MeV})^4$ inferred from the formula (2). Also the flavour-singlet axial charge could be related to the slope of the topological susceptibility, i.e. $\chi'(k^2) \equiv \mathrm{d}\chi(k^2)/\mathrm{d}k^2$, via the Shore-Veneziano formula [4] (for a comprehensive review see Ref. [5]) that is given by

$$\frac{a^0(Q^2)}{a^8} = \frac{\sqrt{6}}{f_\pi}\sqrt{-\chi'_{\mathrm{full}}(0)}\bigg|_{Q^2}. \tag{3}$$

The numerical result deduced from the QCD sum rule proved to show acceptable agreement with the EMC-SMC experiments [6]; the QCD sum rule calculation [4] renders $\chi'_{\mathrm{full}}(0)|_{Q^2=10\,\mathrm{GeV}^2} = -(23.2\ \mathrm{MeV})^2$, that leads to $a^0(Q^2 = 10\ \mathrm{GeV}^2) = 0.353$ and gives $\Gamma_1^\mathrm{p}(Q^2 = 10\ \mathrm{GeV}^2) = 0.143$ for the first moment of the polarised proton structure function.

It is worth while noting the difference between $\chi_{\mathrm{pure}}$ and $\chi_{\mathrm{full}}$ before going on our discussion. $\chi_{\mathrm{pure}}$ is the topological susceptibility evaluated within the pure gluonic theory, while $\chi_{\mathrm{full}}$ is that of the full QCD. The difference is whether the contribution from the fermionic matter fields is contained or not. Once the theory has a massless fermion, the vacuum becomes entirely independent of the topological theta angle because an arbitrary $\mathrm{U_A}(1)$ transformation on the massless fermionic fields yields arbitrary shift on the theta angle. As a result $\chi_{\mathrm{full}}(0) = 0$ holds exactly in the presence of any massless fermion. In fact non-vanishing $\chi_{\mathrm{full}}(0)$ is attributed to the finite current quark mass. The lattice calculation in full QCD gives $\chi_{\mathrm{full}}(0) \sim (164\,\mathrm{MeV})^4$ in the region where the current quark mass is around $20\,\mathrm{MeV}$ with two flavours [7]. The numerical value of $\chi_{\mathrm{full}}(0)$ is rather close to that of $\chi_{\mathrm{pure}}(0)$. We would say, however, that this coincidence has no convincing validity *a priori*. To make this point more articulate, let us take a view of the slope of the topological susceptibility. In the case of the pure gluonic theory $\chi'_{\mathrm{pure}}(0) \sim (8\,\mathrm{MeV})^2$ is obtained from the QCD sum rule [8]; meanwhile, the preliminary lattice calculation [9] gives $\chi'_{\mathrm{full}}(0) \sim -(19\,\mathrm{MeV})^2$,

*E-mail: fuku@nt1.c.u-tokyo.ac.jp
†E-mail: konishi@nt1.c.u-tokyo.ac.jp
‡E-mail: ohta@nt1.c.u-tokyo.ac.jp

which is consistent with the result from the QCD sum rule as is quoted above. Not only the magnitude but also the sign is different between $\chi'_{\rm pure}(0)$ and $\chi'_{\rm full}(0)$. In other words the sign of the slope of the topological susceptibility would tell us which one is regarded as $\chi'_{\rm NJL}(0)$.

*2. The slope of the topological susceptibility in the NJL model.* In our previous work [10] we employed the topological susceptibility to adjust the coupling strength of the 't Hooft interaction within the three flavour Nambu–Jona-Lasinio (NJL) model. Although the results reproduce the desirable tendencies, there are subtleties in the physical interpretation of the obtained topological susceptibility in the NJL model ($\chi_{\rm NJL}(k^2)$) where the gluonic contributions are considered to be integrated out. One of the purposes in this letter is to clarify whether $\chi_{\rm NJL}(k^2)$ should be regarded as $\chi_{\rm full}(k^2)$ or as $\chi_{\rm pure}(k^2)$. The topological susceptibility itself is not sensitive to the distinction as mentioned before but the slope of the topological susceptibility instead is suitable for this aim. Thus we calculate $\chi_{\rm NJL}(k^2)$ with the conventional parameter set in Ref. [11];

$$m_{\rm u} = m_{\rm d} = 5.5\ \text{MeV}, \quad m_{\rm s} = 135.7\ \text{MeV}, \quad \Lambda = 631.4\ \text{MeV}$$
$$G\Lambda^2 = 1.835, \quad K\Lambda^5 = 9.29$$

for the NJL interaction Lagrangian we adopt here,

$$\mathcal{L}_4 = G\sum_{a=0}^{8}\left[(\bar\psi\lambda^a\psi)^2 + (\bar\psi{\rm i}\gamma_5\lambda^a\psi)^2\right],$$
$$\mathcal{L}_6 = -K\left[\det\bar\psi(1+\gamma_5)\psi + \det\bar\psi(1-\gamma_5)\psi\right],$$

where $\lambda^a$ is the Gell-Mann matrix in the flavour space with $\lambda^0 = \sqrt{2/3}\,\text{diag}(1,1,1)$. The determinants are with respect to the flavour indices. The actual procedure to calculate $\chi_{\rm NJL}(k^2)$ is almost the same as that demonstrated in the previous work [10]. The only alteration is the momentum insertion in the expression of the topological susceptibility as shown in Fig. 1 diagrammatically. In order to reach $\chi'(k^2)$ we should take care in the treatment of $k^2$ because the temporal (thermal) direction becomes distinctive at finite temperature. In order to retain the consistency with the calculation of $g_{\pi q\bar q}$ at finite temperature, the differentiation with respect to a squared four-momentum $k^2$ is replaced by that with respect to the fourth component $k_4^2$, which makes no difference at zero temperature due to the Lorentz covariance.

Then the result achieved in the NJL model at zero temperature [1] is

$$\chi'_{\rm NJL}(0) = -(20.8\ \text{MeV})^2, \tag{4}$$

which implies that $\chi_{\rm NJL}(0)$ should be regarded as $\chi_{\rm full}(0)$ rather than as $\chi_{\rm pure}(0)$. In fact $\chi_{\rm NJL}(0)$ calculated with the above parameters takes the value of $(166\ \text{MeV})^4$ [10], which is close to the full topological susceptibility achieved in the lattice simulation as is quoted above. It follows that we should have taken $\chi_{\rm full}(0)$ to adjust the strength of the 't Hooft interaction, $K$. Nevertheless, since $\chi_{\rm full}(0)$ with massive quarks is known to behave similarly as $\chi_{\rm pure}(0)$ [7], the essential conclusion that $K$ does not have to get smaller in order to reproduce the temperature dependence of $\chi_{\rm pure}(0)$ would not be amended at all. So we fix the value of $K$ at zero temperature. The resultant behaviour of $\chi'_{\rm NJL}(0)$ at finite temperature is depicted by the solid curve in Fig. 2.

In order to observe the relevance upon the anomalous suppression of the flavour-singlet axial charge it is necessary to handle $\chi'_{\rm NJL}(0)$ at the experimental scale $Q^2 = 10\ \text{GeV}^2$. We can immediately change the scale by using the solution of the renormalisation group equation at the one-loop order [4];

$$\chi'(0)|_{Q^2} = \chi'(0)|_{Q_0^2}\exp\left(\frac{16}{\beta_1^2\ln(Q^2/\Lambda^2)} - \frac{16}{\beta_1^2\ln(Q_0^2/\Lambda^2)}\right), \tag{5}$$

where $\beta_1 \equiv -\frac{1}{2}(11 - \frac{2}{3}N_{\rm f})$ with $N_{\rm f} = 3$ and $\Lambda = 350$ MeV is the QCD scale parameter. The solution is so sensitive to the initial scale parameter $Q_0^2$ (the NJL scale) that the quantitative results would yield a rough estimate. The dotted curve in Fig. 2 shows the scaled solution for $Q_0^2 = 1\ \text{GeV}^2$. At zero temperature the scaled $\chi'_{\rm NJL}(0)$ is

$$\chi'_{\rm NJL}(0)|_{Q^2=10\,\text{GeV}} = -(18.9\ \text{MeV})^2, \tag{6}$$

which leads to $a^0(Q^2 = 10\ \text{GeV}^2) = 0.288$ and $\Gamma_1^{\rm p}(Q^2 = 10\ \text{GeV}^2) = 0.130$. Comparing with the experimental results from EMC ($\Gamma_1^{\rm p}(Q^2 = 10.7\ \text{GeV}^2) = 0.126$), we reckon that the description by the NJL model furnished with the 't Hooft interaction is quite acceptable, at least in the regime relevant to the $\text{U}_{\rm A}(1)$ sector.

---

[1] For further details on the calculation see Ref. [10] or our forthcoming full paper.

*3. Summary and conclusions.* We have evaluated the slope of the topological susceptibility within the framework of the three flavour NJL model. Its sign implies that $\chi_{\rm NJL}(k^2)$ should be regarded as $\chi_{\rm full}(k^2)$ rather than as $\chi_{\rm pure}(k^2)$. Also we have found that the actual value of $\chi'_{\rm NJL}(0)$ is quite consistent with the Shore-Veneziano formula (3). It is the indirect evidence not only for the validity of the interpretation of $\chi_{\rm NJL}(k^2)$ as $\chi_{\rm full}(k^2)$ but also for the reliability of the Shore-Veneziano formula, that is not enough tested yet. So the results we have presented in this letter is an additional new support for the credibility of the formula (3). Then we have extended our results to the finite temperature case. $\chi'_{\rm NJL}(0)$ becomes smaller as the temperature rises and eventually turns negative beyond $T \simeq 150$ MeV. Thus the Shore-Veneziano formula must break down somewhere at finite temperature. The speed of dropping $\chi'_{\rm NJL}(0)$ is much faster than that of the pion decay constant $f_\pi$ in the right-hand-side of Eq. (3). Accordingly the ratio of the axial charges would decrease at higher temperature as shown in Fig. 3. This behaviour contradicts our naive expectation; if the effective restoration of the $\mathrm{U_A}(1)$ symmetry occurs at sufficiently high temperature, the OZI approximation begins to work well to give $a^0/a^8 \to 1$. As far as we know, the finite temperature behaviour so far is investigated neither in experiment nor in the lattice simulation. We suppose that our prediction shown in Fig. 2 should be verified in other ways, say, by the lattice calculation. We are making progress in further discussions on the finite temperature behaviour from the phenomenological point of view.

## ACKNOWLEDGMENTS

One of the authors (K. F. ) thanks Carl Shakin for his comments on the full topological susceptibility.


[1] S. L. Adler, Phys. Rev. 177 (1969) 2426;
J. S. Bell and R. Jackiw, Nuovo Cim. 60 (1969) 47.

[2] E. Witten, Nucl. Phys. **B156** (1979) 269;
G. Veneziano, Nucl. Phys. **B159** (1979) 213.

[3] B. Allés, M. D'Elia and A. Di Giacomo, Nucl. Phys. **B494** (1997) 281.

[4] G. M. Shore and G. Veneziano, Phys. Lett. **B244** (1990) 75;
S. Narison, G. M. Shore and G. Veneziano, Nucl. Phys. **B433** (1995) 209.

[5] G. M. Shore, SWAT-98-209, CERN-TH-98-374, e-print: `\protect\vrule width0pt\protect\href{http://arXiv.org/abs/hep-ph/9812354}{hep-ph/9812354}`.

[6] J. Ashman et al.(EM Collab.), Phys. Lett. **B206** (1988) 364;
D. Adams et al.(SM Collab.), Phys. Lett. **B329** (1994) 399;
D. Adams et al.(SM Collab.), Phys. Lett. **B339** (1994) 332, Erratum.

[7] B. Allés, M. D'Elia and A. Di Giacomo, Phys. Lett. **B483** (2000) 139.

[8] S. Narison, Phys. Lett. **B255** (1991) 101.

[9] G. Boyd, B. Allés, M. D'Elia and A. Di Giacomo, IFUM-596-FT, IFUP-55-97, e-print: `\protect\vrule width0pt\protect\href{http://arXiv.org/abs/hep-lat/9711025}{hep-lat/9711025}`.

[10] K. Fukushima, K. Ohnishi and K. Ohta, Phys. Rev. **C63** 045203.

[11] T. Hatsuda and T. Kunihiro, Phys. Rep. **247** (1994) 221.


FIG. 1. Feynman diagram to be evaluated up to the leading order of $1/N_{\rm c}$ expansion.

FIG. 2. The behaviour of the slope of the topological susceptibility at finite temperature

FIG. 3. The behaviour of the ratio of the axial charges at finite temperature

Figure 1

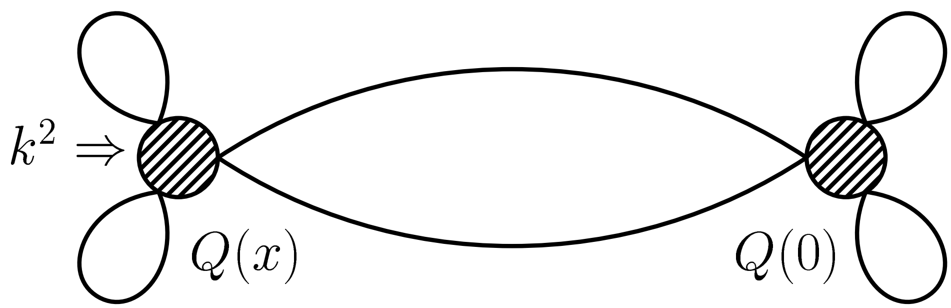
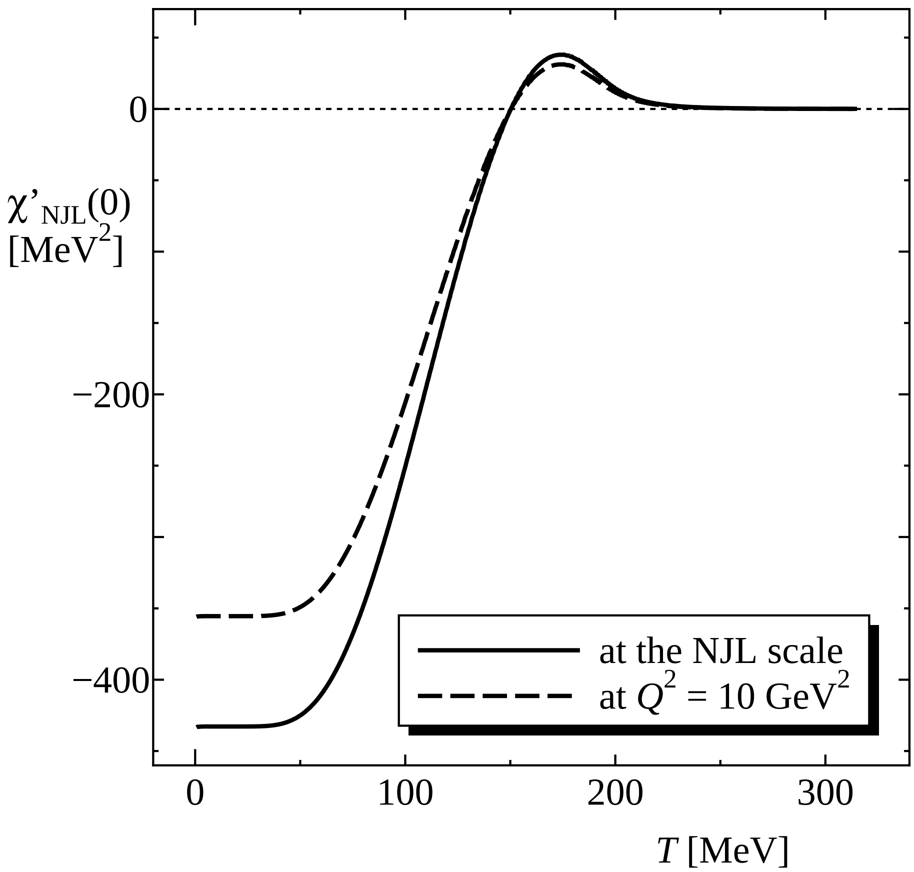


Figure 2



0
−200
−400
χ'NJL(0)
[MeV2]
0
100
200
300
T [MeV]
at the NJL scale
at Q2 = 10 GeV2

Figure 3